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Ramsey Classes of Set Systems

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INTRODUCTION

The purpose of this paper is to prove a strengthening of [4], which gives (practically) the full characterization of Ramsey classes of (ordered) set systems. We also provide a modified proof of the main result of [4]. Let us remark that [4] contains several technical mistakes but they can be corrected by changing details. Although the basic ideas of the proof here and of [4] are the same both differ at several places and we hope that the present proof is more direct and transparent.

We leave out the motivation for this particular research (see e.g., [3, 5, and 6]. Particular cases, which appeared e.g., in [1, 7–9] are simpler results. Presently, there is no other proof for the theorem presented here apart from [4]).

This paper is written in the finite set theory. To state the main result we need some preliminaries. A family $\Delta = (\delta_i; i \in I)$ of natural numbers is called a *type*. A *set system of type* Δ is a pair $(X, \mathscr{M})$, where X is a (totally) ordered set, $\mathscr{M} = (\mathscr{M}_i; i \in I)$ and $M \subseteq X$, $|M| = \delta_i$ for every $M \in \mathscr{M}_i$. The ordering of X is called the *standard ordering* and usually denoted by $\leqslant$. We shall sometimes write explicitly $((X, \leqslant), \mathscr{M})$. Let $A = (X, \mathscr{M})$ is said to be a *subobject* of $B = (Y, \mathscr{N})$, $\mathscr{N} = (\mathscr{N}_i; i \in I)$ if X is a monotone subset of Y (monotone with respect to the standard orderings) and $\mathscr{M}_i = \{M \in \mathscr{N}_i; M \subseteq X\}$ for every $i \in I$.

Denote by Soc(Δ), the class of all set systems of type Δ together with all embeddings. The set of all subobjects of B, which are isomorphic to A, will be denoted by $\binom{B}{A}$. The elements of the set $\binom{B}{A}$ will be also called A-

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subobjects. A set system $(X, \mathscr{M})$ is said to be *irreducible* if, for every $x, y \in X$, there exists an edge $M \in \bigcup \mathscr{M}$ such that $\{x, y\} \subseteq M$.

Let $\mathfrak{a}$ be a family of irreducible set systems of type Δ. Denote by $\mathrm{Soc}(\Delta, \mathfrak{a})$, the subclass of $\mathrm{Soc}(\Delta)$ formed by all those $(Y, \mathscr{N}) \in \mathrm{Soc}(\Delta)$ which satisfy: if $(X, \mathscr{M}) \in \mathrm{Soc}(\Delta)$ and $X \subseteq Y$, $\mathscr{M} \subseteq \mathscr{N}$ (this means $\mathscr{M}_i \subseteq \mathscr{N}_i$ for every $i \in I$), then $(X, \mathscr{M}) \notin \mathfrak{a}$. In other words, $\mathfrak{a}$ is a set of forbidden subsystems for the class $\mathrm{Soc}(\Delta, \mathfrak{a})$. The following is the main result of [4] which will be reproved here.

THEOREM A. *Let Δ be a fixed type. Let $\mathfrak{a}$ be a set of irreducible set systems of type Δ. Then,* $\mathrm{Soc}(\Delta, \mathfrak{a})$ *is a* Ramsey class. *This means the following*: *For every $A, B \in \mathrm{Soc}(\Delta, \mathfrak{a})$ there exists $C \in \mathrm{Soc}(\Delta, \mathfrak{a})$ such that for every coloring c, $\binom{C}{A} \to \{1, 2\}$, there exists $B' \in \binom{C}{B}$ such that c restricted to the set $\binom{B'}{A}$ is a constant.*

Using Theorem A we may give a nearly complete characterization of Ramsey classes of set systems by means of these concepts.

Let $\mathscr{K}$ be a class of set systems of type Δ. Class $\mathscr{K}$ is called *hereditary* if $(X, \mathscr{M}) \in \mathscr{K}$ and $(Y, \mathscr{N})$ satisfies $Y \subseteq X$ and $\mathscr{N} \subseteq \mathscr{M}$, then $(Y, \mathscr{N}) \in \mathscr{K}$.

Class $\mathscr{K}$ is called *order complete* if $((X, \leqslant), \mathscr{M}) \in \mathscr{K}$ and an ordering $\preccurlyeq$ of X coincides with $\leqslant$ on every edge $M \in \bigcup \mathscr{M}$, then, also $((X, \preccurlyeq), \mathscr{M}) \in \mathscr{K}$. Class $\mathscr{K}$ is said to have *unions* if, for every $(X, \mathscr{M})$ and $(X', \mathscr{M}')$ there exists a $(Y, \mathscr{N})$ such that $(X, \mathscr{M})$ and $(X', \mathscr{M}')$ are isomorphic to distinct subobjects of $(Y, \mathscr{N})$. Class $\mathscr{K}$ is said to have *multiplication of points* if, for every $(X, \mathscr{M}) \in \mathscr{K}$ and $x \in X$, $x' \notin X$, the set system $(X \cup \{x'\}, \mathscr{M}')$, where $\mathscr{M}'_i = \mathscr{M}_i \cup \{M - \{x\} \cup \{x'\};\, x \in M \in \mathscr{M}_i\}$ belongs to $\mathscr{K}$.

Now we show the principal result of this paper.

MAIN THEOREM. *Let Δ be a fixed type. Let $\mathscr{K}$ be a class of set systems of type Δ which is hereditary, order complete, and which has unions and multiplication of points. Then, the following two statements are equivalent*:

(1) *Class $\mathscr{K}$ is a Ramsey class.*

(2) *Class* $\mathscr{K} = \mathrm{Soc}(\Delta, \mathfrak{a})$ *for a class $\mathfrak{a}$ of irreducible set systems.*

Numerous examples show that the above restrictions on $\mathscr{K}$ are, in general, necessary. This result generalizes all previously known examples of Ramsey classes of graphs and set systems (see e.g., [4–6].)

As can be expected, one direction is much easier. It can be seen as follows:

Proof $1 \Rightarrow 2$. As $\mathscr{K}$ is a hereditary class, there exists a class $\mathfrak{a}$ of set systems satisfying $\mathscr{K} = \mathrm{Soc}(\Delta, \mathfrak{a})$. To the contrary, assume that any such $\mathfrak{a}$ contains a nonirreducible set system. In particular, then, there exists a set system $(X, \mathscr{M})$ with the following properties:

(i) $(X, \mathscr{M}) \notin \mathscr{K}$;

(ii) $(X', \mathscr{M}') \in \mathscr{K}$ whenever $X' \subseteq X$, $\mathscr{M}' \subseteq \mathscr{M}$, $X' \neq X$;

(iii) there are distinct vertices $\{x', x''\} \subseteq X$ such that there is no edge $M \in \bigcup \mathscr{M}$ with $\{x', x''\} \subseteq M$.

Let B' (B'', A, respectively) be the set system induced on the set $X \backslash \{x'\}$ ($X \backslash \{x''\}$; $X \backslash \{x', x''\}$, respectively). Explicitly, $B' = (Y', \mathscr{N}')$, where $Y' = X \backslash \{x'\}$, $\mathscr{N}' = (\mathscr{N}'_i; i \in I)$, where $N \in \mathscr{N}'_i$, iff $N \in \mathscr{M}_i$ and $N \subseteq X \backslash \{x'\}$; the standard ordering of B' is the restriction of the standard ordering of X to the set Y'.

Let $B \in \mathscr{K}$ be a set system which contains distinct subobjects, which are isomorphic to B' and B''. As $\mathscr{K}$ is a Ramsey class, there exists a set system C with the following properties:

(a) $C \in \mathscr{K}$;

(b) for every coloring c, $\binom{C}{A} \to \{1, 2\}$, there exists a subobject B_o of C, B_o isomorphic to B, such that c restricted to the set $\binom{B_o}{A}$ is a constant mapping.

This C contains a B' subobject $(Y_1, \mathscr{N}_1)$, and a B'' subobject $(Y_2, \mathscr{N}_2)$, such that the subobject of C induced on $Y_1 \cap Y_2$ is isomorphic to A for, otherwise, we could color A subobjects of C which are contained in a B' subobject of C by 1, and the rest of A subobjects of C by 2. Using multiplication of points it is $Y_1 \neq Y_2$. This means, however, that the subobject of C induced on the set $Y_1 \cup Y_2$ is isomorphic to $((X, \lesssim), \mathscr{M})$ for an ordering $\lesssim$ which satisfies $\lesssim|_M = \leqslant|_M$ for every edge $M \in \mathscr{M}$. Consequently, $((X, \lesssim), \mathscr{M}) \in \mathscr{K}$ and hence, $(X, \mathscr{M}) \in \mathscr{K}$, is a contradiction. ∎

The reverse implication $2 \Rightarrow 1$, is much more difficult. Theorem A states exactly what has to be proved. A proof of Theorem A is the core of [4] and of this paper.

In the next proof, we shall work with several classes of structured set systems and with subobjects of various kinds. Therefore, it is convenient to use the categorical language. It is important that the notion of a Ramsey-type statement may be defined uniformly for any structure, which results with these concepts.

Let $\mathscr{K}$ be a class endowed with subobjects. For objects A, B of $\mathscr{K}$, denote by $\binom{B}{A}$, the set of all subobjects of B which are isomorphic to A.

Let A, B, C be objects of $\mathscr{K}$. Object C is said to be *A Ramsey for B* iff, for every coloring c, $\binom{C}{A} \to \{1, 2\}$ there exists $B' \in \binom{C}{B}$ such that c restricted to the set $\binom{B'}{A}$ is a constant. If for every $B \in \mathscr{K}$, there exists an A Ramsey object $C \in \mathscr{K}$ for B, then $\mathscr{K}$ is said to have *A Ramsey property.*

One immediately observes that $\mathscr{K}$ has A Ramsey property iff, for every positive integer r and $B \in \mathscr{K}$, there exists a $C \in \mathscr{K}$ such that for every coloring c, $\binom{C}{A} \to \{1,..., r\}$, there exists $B' \in \binom{C}{B}$ such that c restricted to $\binom{B'}{A}$ is a constant mapping. Using this concise terminology $\mathscr{K}$ is a *Ramsey class*, if it has A Ramsey property for every object $A \in \mathscr{K}$.

Let us remark that we find it convenient to work not only with subobjects, but, also, with special morphisms which we call embeddings.

This paper has four sections. Section 1 contains the definition of classes $a\mathrm{Part}(\Delta)$, $a\mathrm{Soc}(\Delta)$, and $\omega/a\,\mathrm{Soc}(\Delta, \mathfrak{a})$. These classes provide a convenient structural strengthening of classes $\mathrm{Soc}(\Delta, \mathfrak{a})$. Also, Theorems I and II are stated here. Section 2 contains four easy, auxiliar lemmas. Section 3 contains a proof of Theorem I. This is the main part of the paper. Assuming Theorem I, one can derive Theorem II essentially in the same way. This is shown in Section 4. Theorem II is a generalization of Theorem A.

1. Basic Notions and Notation

A family $\Delta = (\delta_i; i \in I)$ of natural numbers is called a type. Mostly, we shall work with (totally) ordered sets. Ordering of a set usually will be denoted by $\leqslant$. If X is an ordered set, A, B are subsets of X, and if $a < b$ for every $a \in A$ and $b \in B$, then we write $A < B$. The maximal element of an ordered set X, will be denoted by $\max X$.

We set $[i,j] = \{i, i+1,..., j\}$ and $[X]^k = \{Y \subseteq X, |Y| = k\}$. We also write $\{x_1,..., x_n\}_<$, providing that $x_1 < \cdots < x_n$.

Let $\Delta = (\delta_i, i \in I)$, $\delta_i \geqslant 1$, be a fixed type, let $a \geqslant 0$ be fixed. The proof of Theorem A (see Introduction) is based on a suitable combination of two classes which are denoted by $a\mathrm{Part}(\Delta)$ and $a\mathrm{Soc}(\Delta)$. These classes are refinements of the class $\mathrm{Soc}(\Delta)$. The classes $a\mathrm{Part}(\Delta)$ and $a\mathrm{Soc}(\Delta)$ have the same objects but they differ in morphisms.

Objects of $a\mathrm{Part}(\Delta)$ *and* $a\mathrm{Soc}(\Delta)$

Pairs $((X_i)_{i=0}^a, \mathscr{M})$, where:

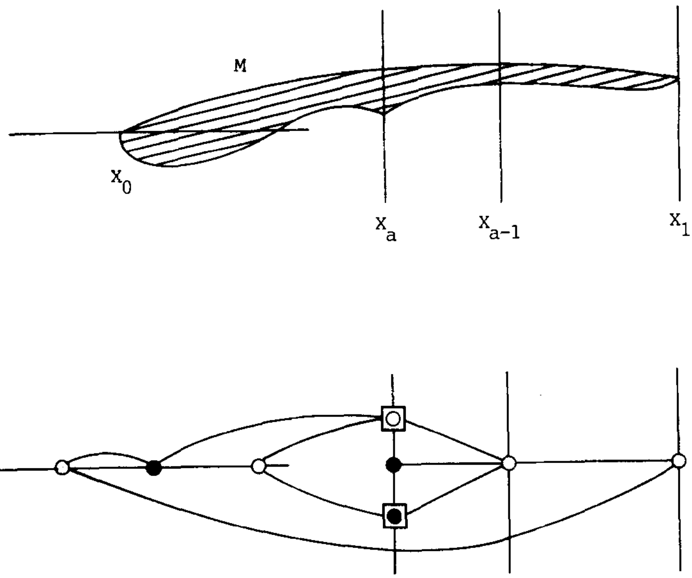
(1) $\bigcup (X_i; i = 0,..., a) = X$ is an ordered set satisfying $X_0 < X_a < X_{a-1} < \cdots < X_1$;

(2) $\mathscr{M} = (\mathscr{M}_i; i \in I)$, where $\mathscr{M}_i \subseteq [X]^{\delta_i}$;

(3) $|M \cap X_i| \leqslant 1$ for every $M \in \mathscr{M}$ and every $0 < i \leqslant a$.

Figure 1 may be helpful.

Objects of $a\mathrm{Part}(\Delta)$ and $a\mathrm{Soc}(\Delta)$ will be denoted by capital letters $A, B,...$. For an object $A = ((X_i)_{i=0}^a, \mathscr{M})$ we write $V(A) = \bigcup (X_i, i = 0,..., a)$, $V_i(A) = X_i$, $E(A) = \bigcup \mathscr{M}$, $E_i(A) = \mathscr{M}_i$. Elements of $V(A)$ are called vertices,



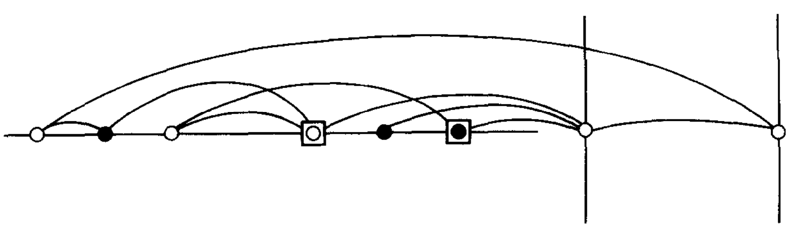
A ∈ 3 Part ((1,1,2))

FIGURE 1

elements of $E(A)$ are edges of A. For the sake of brevity we sometimes write $(X_i)_0^a$ instead of $(X_i)_{i=0}^a$.

An Important Convention

Let $A = ((X_i)_0^a, \mathscr{M}) \in a\mathrm{Part}(\Delta)$. Let b satisfy $0 \leqslant b \leqslant a$. A will be sometimes considered as an object of $b\mathrm{Part}(\Delta)$. If we write $A \in b\mathrm{Part}(\Delta)$, then we mean the object $((X_i')_0^b, \mathscr{M})$, where $X_i' = X_i$ for $i > 0$, and $X_0' = X_0 \cup X_a \cup \cdots \cup X_{b+1}$. The object $A \in 3\mathrm{Part}((1, 1, 2))$ depicted in Fig. 1 considered as object $2\mathrm{Part}((1, 1, 2))$ is shown in Fig. 2.



A ∈ 2 Part ((1,1,2))

FIGURE 2

Morphisms of $a\mathrm{Part}(\Delta)$ *(called embeddings)*

Let

$$A = ((X_i)_0^a, \mathcal{M}), \qquad \mathcal{M} = (\mathcal{M}_i; i \in I),$$

$$B = ((Y_i)_0^a, \mathcal{N}), \qquad \mathcal{N} = (\mathcal{N}_i; i \in I),$$

be objects of $a\mathrm{Part}(\Delta)$. We say that a mapping $f\colon V(A) \to V(B)$ is an *embedding in* $a\mathrm{Part}(\Delta)$ if,

(1) f is a monotone 1–1 mapping (with respect to the standard orderings);

(2) $f(X_i) \subseteq Y_i$ for $i = 0, \dots, a$;

(3) $M \in \mathcal{M}_i \Rightarrow f(M) \in \mathcal{N}_i$ for every $i \in I$;

(4) $f(M) \in \mathcal{N}_i, f(M) \cap Y_0 \neq \emptyset \Rightarrow M \in \mathcal{M}_i$ for every $i \in I$.

If f satisfies conditions (1)–(3) only, then it is said to be a *monomorphism in* $a\mathrm{Part}(\Delta)$.

Morphisms of $a\mathrm{Soc}(\Delta)$ *(called embeddings)*

Let A, B be as above. We say that a mapping $f\colon V(A) \to V(B)$ is an *embedding in* $a\mathrm{Soc}(\Delta)$ if it satisfies conditions (1)–(3) and

(4)′ $f(M) \in \mathcal{N}_i \Rightarrow M \in \mathcal{M}_i$ for every $i \in I$.

Obviously, every embedding in $a\mathrm{Soc}(\Delta)$ is an embedding in $a\mathrm{Part}(\Delta)$ but not conversely. *Isomorphisms*, defined naturally as invertible embeddings, coincide both in $a\mathrm{Part}(\Delta)$ and $a\mathrm{Soc}(\Delta)$. The isomorphisms sign is $\simeq$.

Let A, A', B be objects of $a\mathrm{Part}(\Delta)$ and $a\mathrm{Soc}(\Delta)$, respectively. Assume $V(A') \subseteq V(B)$, $A' \simeq A$. If the inclusion $V(A') \subseteq V(B)$ is an embedding in $a\mathrm{Part}(\Delta)$ and $a\mathrm{Soc}(\Delta)$, respectively, then A' is called an *A subobject of B* in $a\mathrm{Part}(\Delta)$ and in $a\mathrm{Soc}(\Delta)$, respectively. The set of all A subobjects of B will be denoted by $\binom{B}{A}$. From the context it always will be clear whether this symbol applies to the category $a\mathrm{Part}(\Delta)$ or $a\mathrm{Soc}(\Delta)$. If the inclusion $V(A') \subseteq V(B)$ is a monomorphism, then A' is called a *weak A subobject of* B. The set of all weak A subobjects will be denoted by $\binom{B}{A}_w$. Various types of subobjects are illustrated in Fig. 3 for the case $\Delta = ((2))$ (i.e., graphs).

Classes $a\mathrm{Part}(\Delta)$ and $a\mathrm{Soc}(\Delta)$ are convenient refinements of the class $\mathrm{Soc}(\Delta)$, it is $0\mathrm{Part}(\Delta) = 0\mathrm{Soc}(\Delta) = \mathrm{Soc}(\Delta)$. We use this to prove Theorem A (see Introduction).

THEOREM I. *For every $a \geq 0$ and every $A \in \mathrm{Soc}(\Delta)$ with $a \leq |A|$ the class $a\mathrm{Part}(\Delta)$ has A_a Ramsey property. Explicitly, this means the following: For every $B \in a\mathrm{Part}(\Delta)$, there exists $C \in a\mathrm{Part}(\Delta)$ such that for every coloring c, $\binom{C}{A_a} \to \{1, 2\}$ there exists $B' \in \binom{C}{B}$ such that c restricted to the set $\binom{B'}{A_a}$ is a constant.*

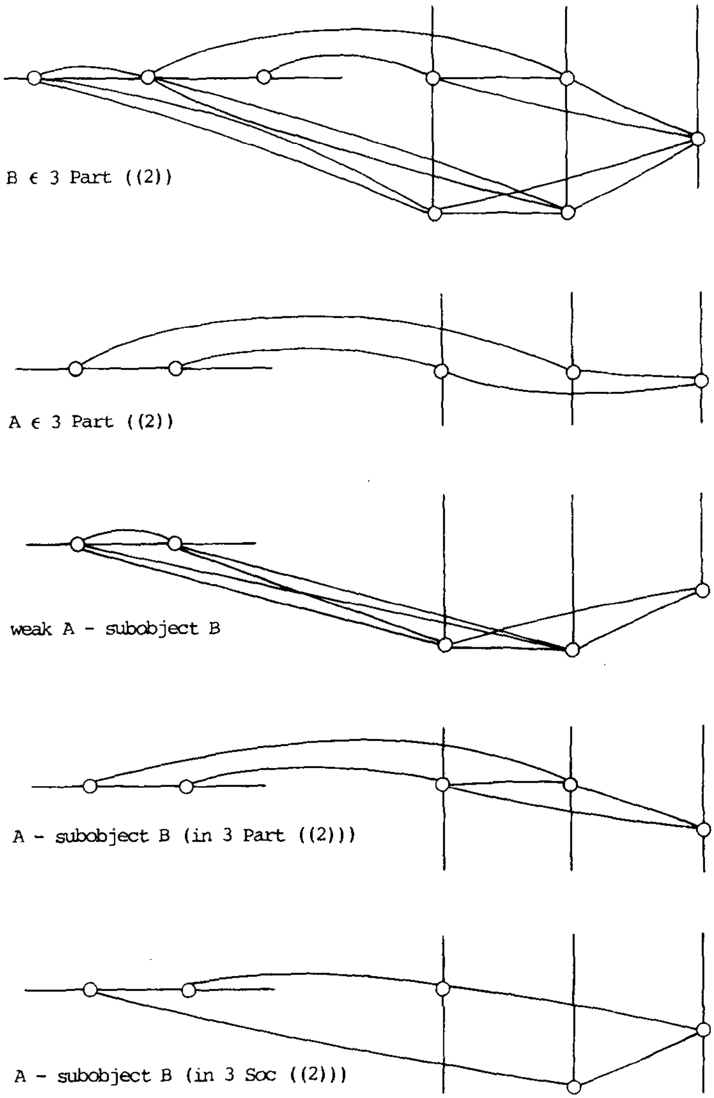
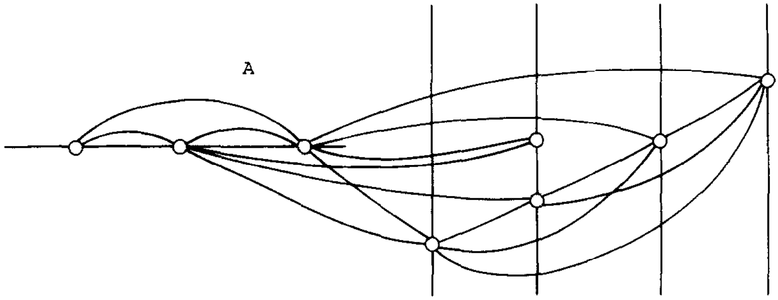

B ∈ 3 Part ((2))
A ∈ 3 Part ((2))
weak A - subobject B
A - subobject B (in 3 Part ((2)))
A - subobject B (in 3 Soc ((2)))

FIGURE 3



$A \notin \frac{\omega}{4}$ Soc $((2), \{K_3\})$ for

$\omega = \phi$, $\{4\}$, $\{3\}$, $\{3,4\}$, $\{2,4\}$, $\{1,4\}$, $\{1,2\}$

FIGURE 4

Here, A_a is defined as follows: Given $A = (X, \mathscr{M}) \in \mathrm{Soc}(\Delta)$, $X = \{x_1, ..., x_m\}_<$, we put $A_a = ((X_i)_0^a, \mathscr{M})$ (for an $a \leqslant m$), where $X_0 = \{x_1, ..., x_{m-a}\}$, $X_i = \{x_{m-i+1}\}$, $i \in [1, a]$. It is $A_a \in a\mathrm{Part}(\Delta)$ and $A_0 = A$.

THEOREM II. *Let* $\mathfrak{a}$ *be a class of irreducible objects of* $\mathrm{Soc}(\Delta)$. *For every* $a \geqslant 0$, $\omega \subseteq [1, a]$ *and every* $A \in \mathrm{Soc}(\Delta)$ *with* $a \leqslant |A|$, *the class* $(\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$ *has* A_a *Ramsey property.*

Here, $(\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$ is a subclass of $a\mathrm{Soc}(\Delta)$ induced by all those objects $B = ((X_i)_0^a, \mathscr{M}) \in a\mathrm{Soc}(\Delta)$ for which there exists no $A \in \mathfrak{a}$ and $A' \in \binom{B}{A}_w$ such that $V(A') \cap V_i(B) \neq \varnothing$, iff $i \in \omega \cup \{0\}$. Classes $(\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$ are convenient refinements of classes $\mathrm{Soc}(\Delta, \mathfrak{a})$. This is illustrated in Fig. 4 for a simple case $\Delta = ((2))$ and $\mathfrak{a} = \{K_3\}$.

Putting $a = 0$ in Theorem II, we get $(\phi/a)\,\mathrm{Soc}(\Delta, \mathfrak{a}) = \mathrm{Soc}(\Delta, \mathfrak{a})$ has $A_0 = A$ Ramsey property for every $A \in \mathrm{Soc}(\Delta, \mathfrak{a})$. This proves Theorem A and, consequently, the main theorem.

Remark. Actually, for every Δ, $\mathfrak{a}$ and a, the classes $a\mathrm{Part}(\Delta)$, $a\mathrm{Soc}(\Delta)$ and $(\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$ are Ramsey classes. It is easy to modify the next proof to get this slightly more general result.

2. LEMMAS

Let $a \geqslant 0$, $\Delta = (\delta_i, i \in I)$ be fixed.

LEMMA I (Singleton lemma). *Classes* $a\mathrm{Part}(\Delta)$, $a\mathrm{Soc}(\Delta)$ *and* $(\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$ *have* A *Ramsey property for every* $A \in a\mathrm{Soc}(\Delta)$, $|V(A)| = 1$, $\omega \subseteq [1, a]$, $\mathfrak{a}$ *a set of irreducible objects of* $\mathrm{Soc}(\Delta)$.

LEMMA II (Good coloring lemma). *For $A_a \in a\mathrm{Part}(\Delta)$ these two statements are equivalent.*

(1) $a\mathrm{Part}(\Delta)$ ($(\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$, *respectively*) *has A_a Ramsey property*;

(2) $a\mathrm{Part}(\Delta)$ ($(\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$, *respectively*) *has good A_a Ramsey property.*

Here, $a\mathrm{Part}(\Delta)$ ($(\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$, respectively) has *good A_a Ramsey property*, iff for every $B \in a\mathrm{Part}(\Delta)$ ($B \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$, respectively) there exists $C \in a\mathrm{Part}(\Delta)$ ($C \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$, respectively) such that for every coloring $c\colon \binom{C}{A_a} \to \{1, 2\}$ there exists $B' \in \binom{C}{B}$ and a mapping $\tilde{c}\colon V_0(B') \to \{1, 2\}$ such that $c(A') = \tilde{c}(\max V_0(A'))$ for every $A' \in \binom{B'}{A_a}$ with $V_0(A') \neq \varnothing$. We also say that B' has a good coloring.

LEMMA III (Induced lemma). *Assume that* $\mathrm{Soc}(\Delta)$ *has A Ramsey property. Then,* $a\mathrm{Soc}(\Delta)$ *has A_a Ramsey property for every $a \geqslant 0$.*

LEMMA IV (Weak partite lemma). *The class $a\mathrm{Part}(\Delta)$ has A_a Ramsey property for every A with $|V(A)| = a$.*

These lemmas are tailored to suit the proof of Theorem I (given in Section 4). They are simple statements which can be proved by standard arguments, (see e.g., [3, 4]).

Proof of Lemma I. Put $A = ((T_i)_0^a, \mathscr{U})$, $|T_j| = 1$, $T_i = \varnothing$ for $i \neq j$. Let $B = ((X_i)_0^a, \mathscr{M})$ be fixed. Set $X_j' = \bigcup\, (V(A'); A' \in \binom{B}{A})$, $k = |X_j'|$. Observe $X_j' \subseteq X_j$. Let $(Y_j', \mathscr{E})$ be a k graph (i.e., a k uniform hypergraph) with a fixed ordering $\leqslant$ of its vertices such that $(Y_j', \mathscr{E})$ does not contain cycles of lengths 2 and 3, and such that the chromatic number $\chi(Y_j', \mathscr{E}) > 2$ (i.e., we are assuming that any two distinct edges intersect in, at most, one point, and that any three edges which pairwise intersect, have a common point, and the chromatic number means that for every partition $Y_j' = Y_1 \cup Y_2$ there exists an edge, $E \in \mathscr{E}$ such that $E \subseteq Y_i$ for either $i = 1$, or $i = 2$; the existence of a k graph with these properties was proved first in [2]).

Put $\mathscr{E} = \{E_1, \ldots, E_r\}$ and for each $\iota = 1, \ldots, r$ let $\varphi_\iota\colon X_j' \to E_\iota$ be the monotone bijection. Find $C = ((Y_i)_0^a, \mathscr{N}) \in a\mathrm{Part}(\Delta)$ with the following properties:

(i) Y_j contains Y_j' as a monotone subset;

(ii) for every $\iota = 1, \ldots, r$ there exists an embedding $\psi_\iota\colon B \to C$ in $a\mathrm{Part}(\Delta)$ (in $(\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$, respectively) such that $\psi_\iota|_{X_j'} = \varphi_\iota$.

It is easy to construct C as a suitable amalgamation of copies of B with respect to the hypergraph $(Y_j', \mathscr{E})$. This is possible as any two edges of $\mathscr{E}$

intersect in, at most, one vertex. Clearly, we may assume $\psi_\iota(V(B)) \cap \psi_{\iota'}(V(B)) = \varphi_\iota(X_j') \cap \varphi_{\iota'}(X_j')$. It is easy to see that C is A Ramsey for B (as $\bigcup (V(A'); A' \in \binom{C}{A}) = Y_j'$) and that $C \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$, providing that $B \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$. ∎

Proof of Lemma II. One direction is clear. Let $B \in a\mathrm{Part}(\Delta)$ ($B \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$, respectively) be a fixed object. Put $V(A) = \{v_1, \dots, v_n\}_<$ and let $\mathbb{1}$ be the set system of type Δ induced on the set $\{v_{n-a}\}$, $\mathbb{1} \in a\mathrm{Part}(\Delta)$. Let $C_1 \in a\mathrm{Part}(\Delta)$ ($C_1 \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$, respectively) be a $\mathbb{1}$ Ramsey object for B and let $C \in a\mathrm{Part}(\Delta)$ ($C \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$, respectively) be a good A_a Ramsey object for C_1 (the existence of C_1 follows from Lemma I). It follows from the corresponding definitions that C is A_a Ramsey for B. ∎

Proof of Lemma III. Put $A = (U, \mathscr{E})$, $U = \{u_1, \dots, u_b\}_<$. Let $a \geqslant 0$, $B = ((X_i)_0^a, \mathscr{M}) \in a\mathrm{Soc}(\Delta)$ be fixed. Let $C' = (Y, \mathscr{N}') \in \mathrm{Soc}(\Delta)$ be an A Ramsey object for $B' = (V(B), \mathscr{M})$ in the category $\mathrm{Soc}(\Delta)$. Define $C = ((Y_i)_0^a, \mathscr{N})$ as, $Y_\iota = \{(\iota, y); y \in Y\}$, $0 \leqslant \iota \leqslant a$, $\mathscr{N} = (\mathscr{N}_i; i \in I)$, where $\mathscr{N}_i$ is the collection of all sets of the following form: $\{(\iota_j, y_j),\ 1 \leqslant j \leqslant \delta_i\}$, $\{y_1, \dots, y_{\delta_i}\}_< \in N_i'$, $\iota_1 = \cdots = \iota_t < \iota_{\delta_i} < \cdots < \iota_{t+1}$, and $\iota_1 = \iota_2 = \cdots = \iota_t = 0$ providing $t \geqslant 2$. Let the ordering of $\bigcup (Y_\iota; \iota = 0, \dots, a)$ be induced by the ordering of Y and by $Y_0 < Y_a < \cdots < Y_1$. It is $C \in a\mathrm{Soc}(\Delta)$. We prove that C is A_a Ramsey for B.

Let $c\colon \binom{C}{A_a} \to \{1, 2\}$ be a coloring. Define a coloring $c'\colon \binom{C'}{A} \to \{1, 2\}$ as follows: for $A' = (U', \mathscr{E}') \in \binom{C'}{A}$, $U' = \{u_1', \dots, u_b'\}_<$, put $c'(A') = c(A_a')$, where A_a' is the subobject of C induced on the set $\{(0, u_1), \dots, (0, u_{b-a}), (a, u_{b-a+1}), \dots, (1, u_b)\}$. Clearly $A_a' \simeq A_a$.

Then, there exists a B' subobject $\tilde{B}$ of C' such that c' restricted to the set $\binom{\tilde{B}}{A'}$ is a constant. Subobject $\tilde{B}$ is of the form $(\bigcup_{i=0}^a \tilde{X}_i, \tilde{\mathscr{M}})$, $\tilde{X}_0 < \tilde{X}_a < \cdots < \tilde{X}_1$, such that $B = ((\tilde{X}_i)_0^a, \tilde{\mathscr{M}}) \simeq B$ (in $a\mathrm{Soc}(\Delta)$). Let $\bar{B}$ be the subobject of C induced on the set $\bigcup_{i=0}^a (\{i\} \times \tilde{X}_i)$. Clearly, $\bar{B} \simeq B$ and c restricted to $\binom{\bar{B}}{A_a}$ is a constant. ∎

Proof of Lemma IV. Let $B = ((X_i)_0^a, \mathscr{M})$ be given. Put $B' = ((X_i)_1^a, \mathscr{M}')$, where $\mathscr{M}'$ is the restriction of $\mathscr{M}$ to the set $\bigcup (X_i; i = 1, \dots, a)$. Let $(Y_i)_1^a$ be such that, for every coloring of the set $Y_1 \times \cdots \times Y_a$, by two colors, there are subsets $X_i' \subseteq Y_i$, $|X_i| = |X_i'|$, such that the set $X_1' \times \cdots \times X_a'$ is colored by the one color only. (This is the product version of the pigeonhole principle, choose $|Y_1| = 2|X_1| - 1$, $|Y_2| = 2^{|Y_1|}(|X_2| - 1) + 1$, etc.) Set $\mathscr{N}' = (\mathscr{N}_i; i \in I)$, $N \in \mathscr{N}'$ iff $|N| = \delta_i$ and $|N \cap Y_\iota| \leqslant 1$ for every $\iota = 1, \dots, a$. Obviously, A_a subobjects of $((Y_i)_1^a, \mathscr{N}')$ and the set $Y_1 \times \cdots \times Y_a$ are in 1–1 correspondence (see the definition of a subobject in $a\mathrm{Part}(\Delta)$, not true in $a\mathrm{Soc}(\Delta)$).

Define $C = ((Y_i)_0^a, \mathscr{N})$ such that for every set of the form $X_1' \times \cdots \times X_a'$, $X_i' \subseteq Y_i$, and $|X_i'| = |X_i|$, there exists a subobject $\bar{B} \in \binom{C}{B}$ (in $a\mathrm{Part}(\Delta)$) such that $V_i(\bar{B}) = X_i'$, for all $i > 0$. (This follows by a similar amalgamation as in the proof of Lemma I.) It is easy to see that C is A_a Ramsey for B (again, in $a\mathrm{Part}(\Delta)$ only, not in $a\mathrm{Soc}(\Delta)$). ▮

3. Proof of Theorem I

We proceed by induction on $|V(A)|$ and $|V(A)| - a$. For the case $|V(A)| = 1$, see Lemma I. For the case $|V(A)| = a$, see Lemma IV.

Let Δ, A, a be fixed throughout the proof and assume $|V(A)| > 1$, $a < |V(A)|$. For every $A' \in a'\mathrm{Part}(\Delta')$, $|V(A')| < |V(A)|$ we assume that the class $a'\mathrm{Part}(\Delta')$ has A_a' Ramsey property for every $a' \geqslant 0$, Δ'. Particularly, we assume that $0\mathrm{Part}(\Delta') = \mathrm{Soc}(\Delta')$ has A' Ramsey property and, using Lemma III, we assume that $a'\mathrm{Soc}(\Delta')$ has $A_{a'}'$ Ramsey property for every a' and A' with $|V(A')| < |V(A)|$.

We prove that $a\mathrm{Part}(\Delta)$ has A_a Ramsey property. Thus, let $B = ((X_i)_0^a, \mathscr{M}) \in a\mathrm{Part}(\Delta)$ be fixed. Using Lemma II, it suffices to prove the existence of a good A_a Ramsey object $C \in a\mathrm{Part}(\Delta)$ for B. This will be done by induction on $|X_0|$.

The boundary case $X_0 = \varnothing$ is again trivial, as it is $|V(A)| > a$, we may put $B = C$.

Let $|X_0| > 0$. Put $x^* = \max X_0$ and define these systems (the beginning of the proof itself).

$$D = ((Y_i)_0^a, \mathscr{P}), \qquad \mathscr{P} = (\mathscr{P}_i, i \in I),$$
$$E = ((Z_i)_0^a, \mathscr{R}), \qquad \mathscr{R} = (\mathscr{R}_i, i \in I),$$

where

$$Y_i = Z_i = X_i \quad \text{for} \quad i > 0, \quad Y_0 = Z_0 = X_0 \backslash \{x^*\};$$
$$\mathscr{P}_i = \{M \in \mathscr{M}_i, x^* \notin M\},$$
$$\mathscr{R}_i = \{M \backslash \{x^*\}; M \in \mathscr{M}_i, x^* \in M\}.$$

Obviously, $D \in a\mathrm{Part}(\Delta)$, $E \in a\mathrm{Part}(\Delta')$, where $\Delta' = (\max\{0, \delta_i - 1\}; i \in I)$.

We find $C = ((X_i')_0^a, \mathscr{M}') \in a\mathrm{Part}(\Delta)$ such that C is good A_a Ramsey for B in $a\mathrm{Part}(\Delta)$. Object C will be constructed as a convenient product $D'' \times E''$, where D'' and E'' are defined below. The constructions of D'' and E'' use all the inductive hypothesis and are the core of the proof of Theorem I.

I. *Construction of* D''

Using the induction on $|X_0|$ there exists $D' = ((Y_i')_0^a, \mathscr{P}')$ such that D' is a good A_a Ramsey object for D (in $a\mathrm{Part}(\Delta)$).

Set $\binom{D'}{D} = \{D_1, ..., D_r\}$ and $Y'_{a+1} = \{y_1, ..., y_r\}$, we assume without loss of generality $Y'_{a+1} \cap V(D') = \varnothing$. For each D_ι, let $\varphi_\iota : D \to D_\iota$ be the isomorphism in $a\mathrm{Part}(\Delta)$. Define $\mathscr{P}^* = (\mathscr{P}_i^*; i \in I)$ as follows: $N \in \mathscr{P}_i^*$, iff either $N \in \mathscr{P}_i'$ or $N = (\varphi_\iota(M \backslash \{x^*\}) \cup \{y_\iota\}$ for some $x^* \in M \in \mathscr{M}_i$.

Put $D^* = ((Y_i')_0^{a+1}, \mathscr{P}^*)$. Consider D^* as an object of $a\mathrm{Part}(\Delta)$. Clearly, for every coloring of all A_a subobjects of D^* which are disjoint from Y'_{a+1} there exists a B subobject $\bar{B}$ of D^*, $V_0(\bar{B}) \cap Y'_{a+1} = \max V_0(\bar{B})$, such that the coloring c restricted to the set of all A_a subobjects of $\bar{B}$ which are disjoint from Y'_{a+1} is a good coloring.

Now, consider D^* as an object of $(a+1)\,\mathrm{Part}(\Delta)$ and use the induction hypothesis on a to get an object $D'' = ((Y_i'')_0^{a+1}, \mathscr{P}'') \in (a+1)\,\mathrm{Part}(\Delta)$ which is A_{a+1} Ramsey for D^* in $(a+1)\,\mathrm{Part}(\Delta)$ with respect to colorings (of $\binom{D''}{A_{a+1}}$) by means of m colors, m is a large number to be specified below.

II. *Construction of* E''

Put explicitly $A = (U, \mathscr{E})$, $\mathscr{E} = (\mathscr{E}_i; i \in I)$, $U = \{u_1, ..., u_b\}_<$. Define $A_a^\smile \in a\mathrm{Part}(\Delta')$ and $A_{a+1}^* \in (a+1)\,\mathrm{Part}(\Delta)$ as follows:

$$A_a^\smile = ((\{u_1, ..., u_{b-a-1}\}, \{u_b\}, \{u_{b-1}\}, ..., \{u_{b-a+1}\}), \mathscr{E}^\smile),$$
$$\mathscr{E}^\smile = (\mathscr{E}_i^\smile; i \in I),$$
$$A_{a+1}^* = ((\{u_1, ..., u_{b-a-1}\}, \{u_b\}, \{u_{b-1}\}, ..., \{u_{b-a}\}), \mathscr{E}^*),$$
$$\mathscr{E}^* = (\mathscr{E}_i^*; i \in I),$$

where

$$\mathscr{E}_i^\smile = \{T \backslash \{u_{b-a}\}; T \in \mathscr{E}_i, u_{b-a} \in T\}$$
$$\mathscr{E}_i^* = \{T \in \mathscr{E}_i; u_{b-a} \in T\} \cup \{T \subseteq U, |T| = \delta_i, u_{b-a} \notin T\}.$$

Set system $A_a^\smile$ is of a type $\Delta' = (\max\{0, \delta_i - 1\}; i \in I)$. Also put $B^* = ((X_0 \backslash \{x^*\}, \{x^*\}, X_a, ..., X_1), \mathscr{M}^*) \in (a+1)\,\mathrm{Part}(\Delta)$, where $\mathscr{M}_i^* = \mathscr{M}_i \cup \{M \subseteq V(B);\ |M| = \delta_i,\ x^* \notin M$ and $|M \cap X_j| \leqslant 1$ for every $j > 0\}$, $\mathscr{M}^* = (\mathscr{M}_i^*; i \in I)$.

By the induction hypothesis (induction on $|V(A)|$), there exists an $A_a^\smile$ Ramsey object $E' = ((Z_i')_0^a, \mathscr{R}') \in a\mathrm{Soc}(\Delta')$ (in the category $a\mathrm{Soc}(\Delta')$, i.e., with respect to embeddings in $a\mathrm{Soc}(\Delta')$) for E. Define the object $E^* =$

$((Z_i^*)_0^{a+1}, \mathscr{R}^*) \in (a+1)\operatorname{Soc}(\Delta)$ as follows: $Z_i^* = Z_i'$ for $0 \leqslant i \leqslant a$, $Z_{a+1}^* = \{z^*\}$ (we assume $z^* \notin V(E')$). Put $\mathscr{R}^* = (\mathscr{R}_i^*; i \in I)$, where $N \in \mathscr{R}_i^*$, iff either $N = M \cup \{z^*\}$ for some $M \in \mathscr{R}_i'$, or $N = \{z^*\}$ in the case $\{x^*\} \in \mathscr{M}_i$, or $|N| = \delta_i$, $z^* \notin N$, and $|N \cap Z_j| \leqslant 1$ for all $j > 0$. Clearly, for every coloring c, of $\binom{E^*}{A_{a+1}^*}$ considered in $(a+1)\operatorname{Soc}(\Delta)$ there exists $\bar{B} \in \binom{E^*}{B^*}$ (in $(a+1)\operatorname{Soc}(\Delta)$) such that c restricted to $\binom{\bar{B}}{A_{a+1}^*}$ is a constant.

Let Z_i'', $i = 0, \ldots, a$ be sets such that for every coloring c of the set $[Z_0'']^{b-a} \times Z_a'' \times \cdots \times Z_1''$ by means of n colors (n is a large number to be specified later), there exist sets $\bar{Z}_i$, $\bar{Z}_i \subseteq Z_i''$, $|\bar{Z}_i| = |Z_i^*|$, and $i = 0, \ldots, a$ such that c, restricted to the set $[\bar{Z}_0]^{b-a} \times \bar{Z}_a \times \cdots \times \bar{Z}_1$ is a constant. The existence of sets Z_i'' follows from the classical Ramsey theorem by a standard *product* argument.

Let $\mathscr{S} = \{S_1, \ldots, S_r\}$ be the list of all sequences $(\bar{Z}_0, \bar{Z}_a, \ldots, \bar{Z}_1)$, $\bar{Z}_i \subseteq Z_i''$, $|\bar{Z}_i| = |Z_i^*|$. Let z_i^*, $i = 1, \ldots, r$, be distinct elements. Choose an ordering such that $Z_0'' < Z_{a+1}'' < Z_a'' < \cdots < Z_1''$, where $Z_{a+1}'' = \{z_i^*; i = 1, \ldots, r\}$. For each $S_\iota = (\bar{Z}_0, \bar{Z}_a, \ldots, \bar{Z}_1)$, let φ_ι be the monotone bijection $\bigcup (Z_i^*; i = 0, \ldots, a) \to \bigcup (\bar{Z}_i; i = 0, \ldots, a)$.

Finally, define $E'' = ((Z_i'')_0^{a+1}, \mathscr{R}'') \in (a+1)\operatorname{Part}(\Delta)$ as $\mathscr{R}'' = (\mathscr{R}_i''; i \in I)$, where $N \in \mathscr{R}_i''$, iff either there exists $M \in \mathscr{R}_i^*$, $z^* \in M$, and $\iota \in [1, r]$ such that $N = \varphi_\iota(M \backslash \{z^*\}) \cup \{z_\iota^*\}$, or $N \subseteq \bigcup (Z_i''; i = 0, \ldots, a)$, $|N| = \delta_i$, and $|N \cap Z_i| \leqslant 1$ for every $i > 0$.

Object E'' has the following property: For every coloring c (by means of n colors) of the set of all those weak A_a subobjects A of E'', which are disjoint from Z_{a+1}'' (i.e., for which $V_0(A) \cap Z_{a+1}'' = \varnothing$) there exists a weak subobject $\bar{E} = ((\bar{Z}_i)_0^{a+1}, \bar{\mathscr{R}})$ of E'' with the properties:

(i) $\bar{E} \simeq E^*$ in $(a+1)\operatorname{Part}(\Delta)$,

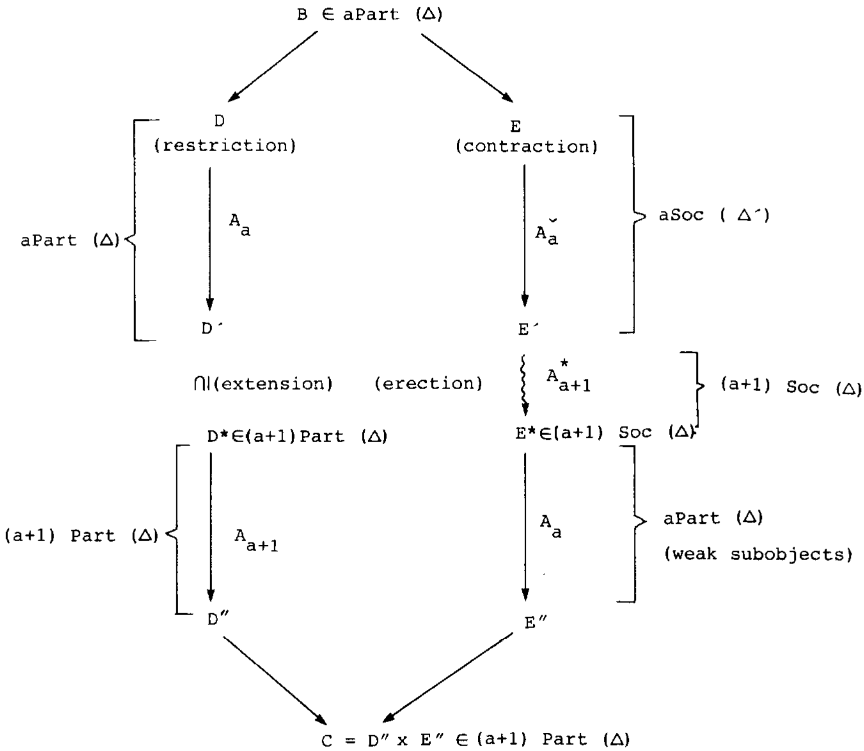
(ii) all the weak A_a subobjects of $\bar{E}$ which are disjoint from Z_{a+1}'' are monochromatic,

(iii) for every $i \in I$ and $N \subseteq \bigcup (\bar{Z}_j; j = 0, \ldots, a+1)$ $N \cap \bar{Z}_{a+1} \neq \varnothing$, holds, $N \in \bar{\mathscr{R}}_i$, iff $N \in \mathscr{R}_i''$.

(That is, $\bar{E}$ is an "induced" subobject with respect to those edges which intersect Z_{a+1}''.) This completes the definition of objects D'' and E''.

Finally, define object $C = ((X_i')_0^{a+1}, \mathscr{M}')$ as follows: $X_i' = Y_i'' \times Z_i''$ for $i = 0, \ldots, a+1$, $\mathscr{M}' = (\mathscr{M}_i'; i \in I)$, $\{(y_\iota, z_\iota); \iota = 1, \ldots, \delta_i\} \in \mathscr{M}_i'$ iff $y_1 < \cdots < y_{\delta_i}$, $z_1 < \cdots < z_{\delta_i}$ and $\{y_\iota; \iota = 1, \ldots, \delta_i\} \in \mathscr{P}_i''$, $\{z_\iota; \iota = 1, \ldots, \delta_i\} \in \mathscr{R}_i''$.

Scheme 1 is a flowchart of the construction of C and may be helpful (the scheme indicates the memberships of objects which are partitioned and the corresponding classes). Let us remark that D is a restriction of B, while E could be called a contraction of B.



SCHEME 1

We prove that C is a good A_a Ramsey for B in $a\mathrm{Part}(\Delta)$. We analyze the set $\binom{C}{A_a}$ first. Define $\binom{C}{A_a}_1$, as the set of all A_a subobjects of C which do not meet $Y''_{a+1} \times Z''_{a+1}$. Analogously, define $\binom{C}{A_a}_2$ as the set of all A_a subobjects of C which meet $Y''_{a+1} \times Z''_{a+1}$. Obviously, $\binom{C}{A_a} = \binom{C}{A_a}_1 \cup \binom{C}{A_a}_2$. Similarly, we define the sets $\binom{D''}{A_a}_1$ and $\binom{E''}{A_a}_{w,1}$ (the latter being the set of all weak subobjects of E'' which do not meet Z''_{a+1}).

Next, we consider the product structure of C and its influence on subobjects of various kinds. Let $\bar{X} = \{(y_1, z_1),\ldots, (y_b, z_b)\}_<$ be the vertex set of a weak A_a subobject of C. Then, of course, $\bar{Y} = \{y_1,\ldots, y_b\}$ is a vertex set of a weak A_a subobject of D'' and $\bar{Z} = \{z_1,\ldots, z_b\}$ is a vertex set of a weak A_a subobject of E''. As a subobject may be identified with its set of vertices, the subobject (determined by) $\bar{X}$ will be denoted by $(\bar{Y}, \bar{Z})$. Moreover, if $\bar{Y}$ induces an A_a subobject of D'' (in $a\mathrm{Part}(\Delta)$), then $\bar{X}$ is an A_a subobject of C (in $a\mathrm{Part}(\Delta)$). In particular, $\binom{C}{A_a}_1$ contains, symbolically, the set $\binom{D''}{A_a}_1 \times \binom{E''}{A_a}_{w,1}$. We put $\mathcal{O}_1 = \binom{D''}{A_a}_1$ and $\mathcal{O}_2 = \binom{E''}{A_a}_{w,1}$, thus $\binom{C}{A_a}_1 \supseteq \mathcal{O}_1 \times \mathcal{O}_2$.

Let $\bar{X}, \bar{Y}, \bar{Z}$ be as above, assume $\bar{X} \cap (Y''_{a+1} \times Z''_{a+1}) \neq \varnothing$. If $\bar{Y}$ is an A_{a+1} subobject in $(a+1)\,\mathrm{Part}(\Delta)$ and $\bar{Z}$ is an A^*_{a+1} subobject in $(a+1)\,\mathrm{Soc}(\Delta)$, then $\bar{X} = (\bar{Y}, \bar{Z})$ is an A_a subobject of C in $a\mathrm{Part}(\Delta)$. Let us verify that the inclusion $\bar{X} \subseteq V(C)$ is an embedding in $a\mathrm{Part}(\Delta)$, (see the definition in Section 1). Clearly, it suffices to check condition 4. Suppose that a subset $M = \{(m_1, m_1'),..., (m_k, m_k')\}$ of $\bar{X}$ is an edge of C. Then, either $M \cap (Y''_{a+1} \times Z''_{a+1}) = \emptyset$ and $\{m_1,..., m_k\} \in E_i(D'')$ implies $\{m_1,..., m_k\} \in E_i(A)$ (using $\bar{Y} \in \binom{D''}{A_{a+1}}$) or $M \cap (Y''_{a+1} \times Z''_{a+1}) \neq \emptyset$ (and possibly $M \cap (Y''_0 \times Z''_0) = \emptyset$), and then, $\{m_1',..., m_k'\} \in E_i(E'')$ implies $\{m_1',..., m_k'\} \in E_i(A)$ (using $\bar{Z} \in \binom{E''}{A^*_{a+1}}$). Thus $\bar{X} \in \binom{C}{A_a}$. Particularly, the set $\binom{C}{A_a}_2$ contains, symbolically, the set $\binom{D''}{A_{a+1}} \times \binom{E''}{A_{a+1}}$. (Observe that $\binom{D''}{A_a}_{w,2} \supseteq \binom{D''}{A_{a+1}}$ and that each A^*_{a+1} subobject of E'' in $(a+1)\,\mathrm{Soc}(\Delta)$ is an A_a subobject in $a\mathrm{Part}(\Delta)$ which meet $V_{a+1}(E'')$.) We put $\mathscr{B}_1 = \binom{D''}{A_{a+1}}$ and $\mathscr{B}_2 = \binom{E''}{A_a}_2$, thus $\binom{C}{A_a}_2$ contains the set $\mathscr{B}_1 \times \mathscr{B}_2$.

Now, we can verify that C is A_a Ramsey for B. Thus let $c\colon \binom{C}{A_a} \to \{1, 2\}$ be a fixed coloring. Choose n such that $n = 2^{|\mathscr{B}_2|}$. Define coloring c'': $\binom{D''}{A_{a+1}} \to \{1, 2\}^{\mathscr{B}_2}$ by $c''(A_1) = (c(A_1, A_2); A_2 \in \mathscr{B}_2)$. Using the properties of D'', there exists $\bar{D}^* \in \binom{D''}{D^*}$ (in $(a+1)\,\mathrm{Part}(\Delta)$) such that c'', restricted to the set $\binom{\bar{D}^*}{A_{a+1}}$, is a constant mapping. Choose m such that $m = 2^{|\mathscr{A}_1'|}$, where $\mathscr{A}_1' = \binom{\bar{D}^*}{A_a} \cap \mathscr{A}_1$. Define the coloring d'': $\binom{E''}{A_a}_{w,1} \to \{1, 2\}^{\mathscr{A}_1'}$ by $d''(A_2) = (c(A_1, A_2), A_1 \in \mathscr{A}_1')$. Using the properties of E'', there exists $\bar{E}^* \in \binom{E''}{E^*}$ such that conditions (i–iii) hold. Define the coloring c': $\binom{\bar{D}^*}{A_a}_1 \to \{1, 2\}$ by $c'(A_1) = i$ iff $c(A_1, A_2) = i$ for every $A_2 \in \binom{\bar{E}^*}{A_a}_{w,1}$. Using the properties of D^*, there exists $\bar{B} \in \binom{\bar{D}^*}{B}$ (in $a\mathrm{Part}(\Delta)$) such that $V_0(\bar{B}) \cap Y''_{a+1} = \{\max V_0(\bar{B})\}$ and such that the color $c'(A_1)$ of an $A_1 \in \binom{B}{A_a}_1$ depends on $\max V_0(A_1)$ only (i.e., $c'(A_1) = \tilde{c}'(\max V_0(A_1))$ for a function $\tilde{c}'$: $V_0(\bar{B}) \to \{1, 2\}$). Finally, define a coloring d': $\binom{\bar{E}^*}{A_a}_2 \to \{1, 2\}$ by $d'(A_2) = i$, iff $c'(A_1, A_2) = i$ for every $A_1 \in \binom{\bar{B}}{A_{a+1}}$. Using the properties of E^*, there exists $\tilde{B} \in \binom{E^*}{B^*}$ (in $(a+1)\,\mathrm{Soc}(\Delta)$) such that d', restricted to the set $\binom{\tilde{B}}{A^*_{a+1}}_2$ (in $a\mathrm{Part}(\Delta)$ which is here the same as $(a+1)\,\mathrm{Soc}(\Delta)$) is a constant mapping (say constant §).

We prove that the subobject of C, induced by $\bar{B}$ and $\tilde{B}$ has the desired properties. To do so, put explicitly, $\bar{B} = ((\bar{Y}_i)_0^{a+1}, \bar{\mathscr{M}})$, $\tilde{B} = ((\bar{Z}_i)_0^{a+1}, \tilde{\mathscr{M}})$, $V(B) = \{x_1,..., x_r\}_<$, $V(\bar{B}) = \{y_1,..., y_r\}_<$, and $V(\tilde{B}) = \{z_1,..., z_r\}_<$.

Let B' be the subobject of C induced on the set $\{(y_1, z_1),..., (y_r, z_r)\}$. Also, set $\mathscr{M}' = E(B')$. We prove:

(1) $B' \simeq B$, and

(2) c restricted to the set $\binom{B'}{A_a}$ is a good coloring.

1. *We prove that $x_i \mapsto (y_i, z_i)$ is an isomorphism*

Consider $M = \{x_\iota; \iota \in \kappa\}_< \subseteq V(B)$. We put $\bar{M} = \{(y_\iota, z_\iota), \iota \in \kappa\}$. It follows from the constructions of the sets $\bar{Y}_i, \bar{Z}_i$, that $x^* \in M$, iff $\bar{M} \cap (\bar{Y}_{a+1} \times \bar{Z}_{a+1}) \neq \emptyset$. If $x^* \notin M$, then using the properties of D'' and D', $\{y_\iota; \iota \in \kappa\} \in \bar{\mathscr{M}}_i$, iff $\{x_\iota; \iota \in \kappa\} \in \mathscr{M}_i$. If $x^* \in M$, then using the properties of E'' and E', $\{z_\iota; \iota \in \kappa\} \in \tilde{\mathscr{M}}_i$, iff $\{x_\iota; \iota \in \kappa\} \in \mathscr{M}_i$.

2. *Definition of* $\tilde{c}$: $V_0(B') \to \{1, 2\}$

Set $\tilde{c}((y_i, z_i)) = \tilde{c}'(y_i)$ for $(y_i, z_i) \neq \max V_0(B')$, $\tilde{c}(\max V_0(B')) = \S$.
Fix $\bar{A} \in \binom{B'}{A_a}$. We prove $c(\bar{A}) = \tilde{c}(\max V_0(\bar{A}))$. There are two possibilities:

(i) $\max V_0(\bar{A}) \in \bar{Y}_{a+1} \times \bar{Z}_{a+1}$,

(ii) $\max V_0(\bar{A}) \notin \bar{Y}_{a+1} \times \bar{Z}_{a+1}$.

Assuming (i), $\bar{A}$ belongs to the set $\binom{D''}{A_{a+1}} \times \binom{E''}{A_a}_2$. Putting $\bar{A} = (A_1, A_2)$, we have that the color of $\bar{A}$ depends on A_2 only and that it is $c(\bar{A}) = d'(A_2) = \S$. Assuming (ii), $\bar{A}$ belongs to the set $\binom{D''}{A_a}_1 \times \binom{E''}{A_a}_{w,1}$. Setting $\bar{A} = (A_1, A_2)$, we have that the color of $\bar{A}$ depends on A_1 only and that it is $c(\bar{A}) = c'(A_1) = \tilde{c}'(\max V_0(A_1))$.

Consequently, c restricted to the set $\binom{B'}{A_a}$, is a good coloring. End of the proof of Theorem I.

4. Proof of Theorem II

Theorem II may be proved using the same pattern as for Theorem I. Therefore, we stress the differences only. The main difference is that the whole proof is carried out in classes $a\mathrm{Soc}(\Delta)$ and $(a+1)\,\mathrm{Soc}(\Delta)$ instead of classes $a\mathrm{Part}(\Delta)$ and $(a+1)\,\mathrm{Part}(\Delta)$. This is possible as we assume the validity of Theorem I and thus, using Lemma III, $a\mathrm{Soc}(\Delta)$ has A_a Ramsey property for every type Δ and $A \in \mathrm{Soc}(\Delta)$.

To prove the A_a Ramsey property of classes $(\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$, we proceed by induction on $|V(A)|$ and $|V(A)| - a$ (i.e., as in Section 3). First, consider the boundary cases. For the case $|V(A)| = 1$ see Lemma I.

The case $|V(A)| = a$ needs some care, it is not covered by Lemma IV. Let $B = ((X_i)_0^a, \mathscr{M}) \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$ be given. Let $B' = ((X_i)_1^a, \mathscr{M}')$ be the subobject of B induced on the set $V(B) \backslash X_0$. Let $C' = ((Y_i)_1^a, \mathscr{N}')$ be an A_a Ramsey object for B' in $a\mathrm{Soc}(\Delta)$ (C' exists by Theorem I and Lemma III). We may assume that $C' \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$ (as $V_0(C') = \varnothing$, see the definition of classes $(\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$). Now, let $C = ((Y_i)_0^a, \mathscr{N})$ be an object of $a\mathrm{Soc}(\Delta)$ such that for every $\bar{B}' \in \binom{C'}{B'}$, there exists $\bar{B} \in \binom{C}{B}$ with $V(\bar{B}) \cap V(C') = V(\bar{B}')$. Object C may be defined as an amalgamation of copies of B with respect to $\binom{C'}{B'}$. As $\mathfrak{a}$ is a class of irreducible objects, (irreducible with respect to an amalgamation) we may suppose $C \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$. Obviously, C is A_a Ramsey for B.

Let us consider the induction step. Let Δ, $\mathfrak{a}$, A, a be fixed. Assume $|V(A)| > 1$ and $a < |V(A)|$. By induction on $|X_0|$, we prove the existence of a good A_a Ramsey object for every object $B = ((X_i)_0^a, \mathscr{M}) \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$. The case $V_0(B) = \varnothing$ is again trivial.

Let $B = ((X_i)_0^a, \mathscr{M}) \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$ be fixed. The following are the induction hypotheses:

(i) the class $(\kappa/a')\,\mathrm{Soc}(\Delta', \mathfrak{a}')$ has $A'_{a'}$ Ramsey property for every type Δ', a class of irreducible set systems $\mathfrak{a}'$, $a' \geqslant 0$, $\kappa \subseteq [1, a']$, $A' \in \mathrm{Soc}(\Delta', \mathfrak{a}')$, $|V(A')| < |V(A)|$;

(ii) $(\kappa/a+1)\,\mathrm{Soc}(\Delta, \mathfrak{a})$ has A_{a+1} Ramsey property for every choice $\kappa \subseteq [1, a+1]$;

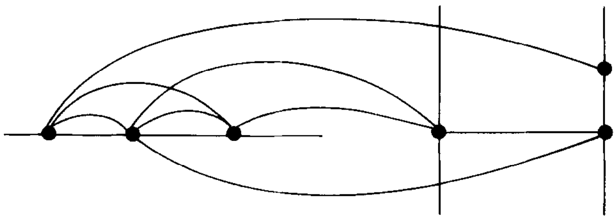
(iii) for every $B' = ((X_i)_0^a, \mathscr{M}') \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$, $|X_0'| < |X_0|$, there exists a good A_a Ramsey object in $(\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$.

We put $x^* = \max X_0$ and we define $D \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$, $D' \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$ a good A_a Ramsey object for D (and D^* an extension of D') as in the proof of Theorem I. We may assume $D^* \in (\omega/(a+1))\,\mathrm{Soc}(\Delta, \mathfrak{a})$ as D^* is a result of an amalgamation. Let $D'' \in (\omega/(a+1))\,\mathrm{Soc}(\Delta, \mathfrak{a})$ be an A_{a+1} Ramsey object for D^* (in $\omega/(a+1)\,\mathrm{Soc}(\Delta, \mathfrak{a})$) for colorings by means of m colors. Thus, the definition of D'' is formally the same as in Section 3. It is the construction of E'' which is slightly different.

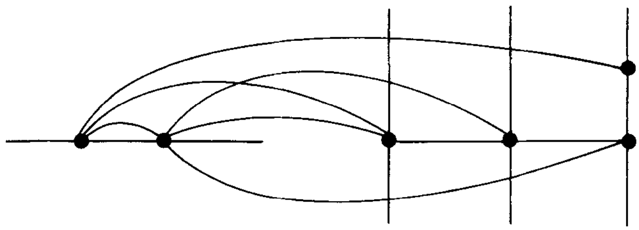
First, define the following operation (the contraction of a vertex of a set system): For a given object $F = ((V_i)_0^a, \mathscr{U}) \in a\mathrm{Soc}(\Delta)$, $v^* = \max V_0$, define the object $F\hat{\ } = ((V_0 \backslash \{v^*\}, V_1, \ldots, V_a), \mathscr{U}\hat{\ })$, $\mathscr{U}\hat{\ } = (\mathscr{U}_\iota; \iota \in I \times \{0, 1\})$, as follows: $\mathscr{U}_{i,0} = \{U \in \mathscr{U}_i; v^* \notin U\}$, $\mathscr{U}_{i,1} = \{U \backslash \{v^*\}; v^* \in U\} \in \mathscr{U}_i\}$ if $\delta_i \geqslant 1$, $\mathscr{U}_{i,1} = \varnothing$, otherwise. Observe that $F\hat{\ } \in a\mathrm{Soc}(\Delta\hat{\ })$, where $\Delta\hat{\ } = (\delta_i; (i, 0) \in I \times \{0\}) \cup (\max\{0, \delta_i - 1\}; (i, 1) \in I \times \{1\})$. Given F as above define also $F^* \in (a+1)\,\mathrm{Soc}(\Delta)$ by $F^* = ((V_0 \backslash \{v^*\}, V_1, \ldots, V_a, \{v^*\}), \mathscr{U})$. Using the definition of classes $(\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$ it is easy to see that $F^* \in (\omega \cup \{a+1\})/(a+1)\,\mathrm{Soc}(\Delta, \mathfrak{a})$, iff $F\hat{\ } \in (\omega/a)\,\mathrm{Soc}(\Delta\hat{\ }, \mathfrak{a}\hat{\ })$. Here, $\mathfrak{a}\hat{\ }$ is the class of all objects $G\hat{\ }$, $G \in \mathfrak{a}$, where $G\hat{\ }$ is the result of the contraction of the $(|V(G)| - |\omega|)$th vertex of G. Figure 5 illustrates this for $\mathfrak{a} = \{K_3\}$, $\Delta = ((2))$.

Let us return to the construction of E''. Put $\omega' = \omega \cup \{a+1\}$. It is $B^* \in (\omega'/(a+1))\,\mathrm{Soc}(\Delta, \mathfrak{a})$ and, therefore, $B\hat{\ } \in (\omega/a)\,\mathrm{Soc}(\Delta\hat{\ }, \mathfrak{a}\hat{\ })$. Let $E' \in (\omega/a)\,\mathrm{Soc}(\Delta\hat{\ }, \mathfrak{a}\hat{\ })$ be an $A\hat{_a}$ Ramsey object for $B\hat{\ }$. Transform E' to $E^* \in (a+1)\,\mathrm{Soc}(\Delta)$ by putting $E^* = ((Z_i^*)_0^{a+1}, \mathscr{M}^*)$, $Z_i^* = Z_i'$, $Z_{a+1}^* = \{z^*\}$, and $\mathscr{R}^* = (\mathscr{R}_i^*; i \in I)$, where $N \in \mathscr{R}_i^*$, iff either $N \in E_{i,0}(E')$ or $N = M \cup \{z^*\}$ for some $M \in E_{i,1}(E')$ or $N = \{z^*\}$ and $\{x^*\} \in \mathscr{M}_i$. It is $E^* \in (\omega'/(a+1))\,\mathrm{Soc}(\Delta, \mathfrak{a})$. Also, E^* is an A_{a+1} Ramsey object for B^* in $(\omega'/(a+1))\,\mathrm{Soc}(\Delta, \mathfrak{a})$. Let E^{**} be the subobject of E^* induced on the set $V(E^*) \backslash V_{a+1}(E^*)$. It is $E^{**} \in a\mathrm{Soc}(\Delta)$ and let $\tilde{E}$ be an A_a Ramsey object for E^{**} in $a\mathrm{Soc}(\Delta)$. Let E'' be an object of $(a+1)\,\mathrm{Soc}(\Delta)$ such that for every $\overline{E^{**}} \in \binom{\tilde{E}}{E^{**}}$ there exists $\overline{E^*} \in \binom{E''}{E^*}$ with $V(\overline{E^*}) \cap V(\tilde{E}) = V(\overline{E^{**}})$. Now, E'' is easy to construct by an amalgamation of copies of E^* with respect to $\binom{\tilde{E}}{E^{**}}$. As $\tilde{E}$ is an A_a Ramsey object for E^{**} in $a\mathrm{Soc}(\Delta)$ and as $E^* \in (\omega'/(a+1))\,\mathrm{Soc}(\Delta, \mathfrak{a})$, we may suppose that $E'' \in (\omega'/(a+1))\,\mathrm{Soc}(\Delta, \mathfrak{a})$.

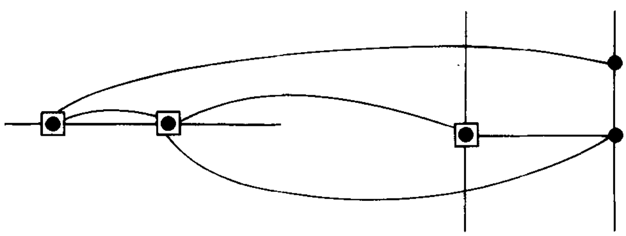
Using D'' and E'', define $C = D'' \times E''$. In the same way as in the Proof of Theorem I, we prove that C is good A_a Ramsey for B. We prove that



$F \in \frac{\{1\}}{2}\ \mathrm{Soc}\,((2),\ )$



$F^* \in \frac{\{1,3\}}{3}\ \mathrm{Soc}\,((2),\ )$



$F^\wedge \in \frac{\{1\}}{2}\ \mathrm{Soc}\,((1,2),\ )$

FIGURE 5

$C \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$. Let $F \in \mathfrak{a}$ and assume that there exists $F' \in \binom{C}{F}$. Put explicitly $V(F') = \{(y_1, z_1),\dots, (y_r, z_r)\}_<$. As F is irreducible, it is $y_1 < \cdots < y_r$ and $z_1 < \cdots < z_r$. Let F_1 be a subobject of D'', induced on the set $\{y_1,\dots, y_r\}$ and let F_2 be a subobject of E'', induced on the set $\{z_1,\dots, z_r\}$. If $V(F') \cap V_{a+1}(C) = \varnothing$, then $V(F_1) \cap V_{a+1}(D'') = \varnothing$ and consequently, it is not true that

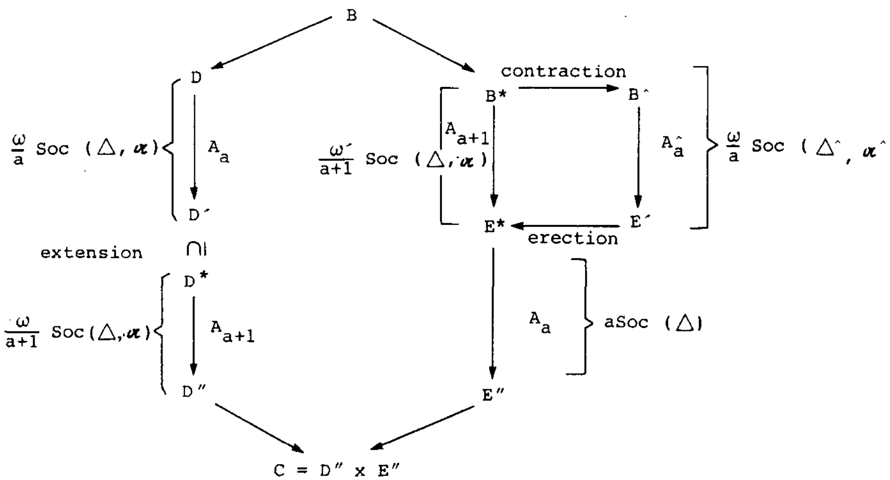
$$V(F_1) \cap V_i(D'') \neq \varnothing \qquad \text{iff} \quad i \in \omega \cup \{0\}$$

(here we use $D'' \in (\omega/(a+1))\,\mathrm{Soc}(\Delta, \mathfrak{a})$). If $V(F') \cap V_{a+1}(C) \neq \varnothing$, then $|V(F_2) \cap V_{a+1}(E'')| = 1$ and, using $E'' \in (\omega'/(a+1))\,\mathrm{Soc}(\Delta, \mathfrak{a})$, we get that it is not true

$$V(F_2) \cap V_i(E'') \neq \varnothing \qquad \text{iff} \quad i \in \omega' \cup \{0\}.$$

This implies that $C \in (\omega/a)\,\mathrm{Soc}(\Delta, \mathfrak{a})$. End of the proof of Theorem II. ∎

Scheme 2 is a flowchart of the construction of C which may be helpful.



SCHEME 2

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